DOWNLOAD PDF CONSEQUENCES OF THE AXIOM OF CHOICE

Chapter 1 : Consequences of the Axiom of Choice

This book, Consequences of the Axiom of Choice, is a comprehensive listing of statements that have been proved in the last years using the axiom of choice. Each consequence, also referred to as a form of the axiom of choice, is assigned a number. Part I is a listing of the forms by number.

It is now a basic assumption used in many parts of mathematics. In fact, assuming AC is equivalent to assuming any of these principles and many others: Given any two sets, one set has cardinality less than or equal to that of the other set -- i. Any vector space over a field F has a basis -- i. If we only consider the case where F is the real line, we obtain a slightly weaker statement; it is not yet known whether this statement is also equivalent to AC. Any product of compact topological spaces is compact. AC has many forms; here is one of the simplest: Let C be a collection of nonempty sets. Then we can choose a member from each set in that collection. In other words, there exists a function f defined on C with the property that, for each set S in the collection, f S is a member of S. The function f is then called a choice function. If C is the collection of all intervals of real numbers with positive, finite lengths, then we can define f S to be the midpoint of the interval S. If C is some more general collection of subsets of the real line, we may be able to define f by using a more complicated rule. However, if C is the collection of all nonempty subsets of the real line, it is not clear how to find a suitable function f. In fact, no one has ever found a suitable function f for this collection C, and there are convincing model-theortic arguments that no one ever will. Of course, to prove this requires a precise definition of "find," etc. The controversy was over how to interpret the words "choose" and "exists" in the axiom: If we follow the constructivists, and "exist" means "find," then the axiom is false, since we cannot find a choice function for the nonempty subsets of the reals. However, most mathematicians give "exists" a much weaker meaning, and they consider the Axiom to be true: To define f S , just arbitrarily "pick any member" of S. In effect, when we accept the Axiom of Choice, this means we are agreeing to the convention that we shall permit ourselves to use a hypothetical choice function f in proofs, as though it "exists" in some sense, even in cases where we cannot give an explicit example of it or an explicit algorithm for it. For an introduction to constructivism, you might take a look at my paper on that subject. The term has rather different, slightly related meanings in advanced mathematics and in mathematics education; I am referring to the former meaning here. To assert that a mathematical object "exists," even when you cannot give an example of it, is a little bit like this: Suppose that one day you go to a football game by yourself. Then you know those people have names, but you cannot give any of those names. The "existence" of f -- or of any mathematical object, even the number "3" -- is purely formal. It does not have the same kind of solidity as your table and your chair; it merely exists in the mental universe of mathematics. Many different mathematical universes are possible. Both possibilities are feasible -- i. However, most "ordinary" mathematicians -- i. Bertrand Russell was more famous for his work in philosophy and political activism, but he was also an accomplished mathematician. Here is my paraphrasing of part of what he said: To choose one sock from each of infinitely many pairs of socks requires the Axiom of Choice, but for shoes the Axiom is not needed. The idea is that the two socks in a pair are identical in appearance, and so we must make an arbitrary choice if we wish to choose one of them. For shoes, we can use an explicit algorithm -- e. Well, if we only have finitely many pairs of socks, then AC is not needed -- we can choose one member of each pair using the definition of "nonempty," and we can repeat an operation finitely many times using the rules of formal logic not discussed here. This is a joke. In the setting of ordinary set theory, all three of those principles are mathematically equivalent -- i. However, human intuition does not always follow what is mathematically correct. For another indication of the controversy that initially surrounded the Axiom of Choice, consider this anecdote recounted by Jan Mycielski in Notices of the AMS vol. Tarski, one of the early great researchers in set theory and logic, proved that AC is equivalent to the statement that any infinite set X has the same cardinality as the Cartesian product X x X. He submitted his article to Comptes Rendus Acad. Both wrote letters rejecting the article. And Lebesgue wrote that an implication between two false statements is of no interest. Tarski said that he never again submitted a paper to the Comptes Rendus. AC permits arbitrary choices from an arbitrary collection of nonempty sets. Some mathematicians have investigated some weakened forms of AC, such as CC Countable Choice , which permits arbitrary choices from a sequence of nonempty sets. DC Dependent Choice , which permits the more general process of selecting arbitrarily from a sequence of nonempty sets where only the first set is specified in advance; each subsequent set of options may depend somehow on the previous choices. This is precisely what is needed for some choice processes in topology and analysis -- e. The full strength of the Axiom of Choice does not seem to be needed for applied mathematics. Some weaker principle such as CC or DC generally would suffice. To see this, consider that any application is based on measurements, but humans can only make finitely many measurements. We can extrapolate and take limits, but usually those limits are sequential, so even in theory we cannot make use of more than countably many measurements. The resulting spaces are separable. Thus, in some sense, nonseparable spaces exist only in the imagination of mathematicians. If we restrict our attention to separable spaces, then much of conventional analysis still works with AC replaced by CC or DC. However, the resulting exposition is then more complicated, and so this route is only followed by a few mathematicians who have strong philosophical leanings against AC. A few pure mathematicians and many applied mathematicians including, e. Although AC simplifies some parts of mathematics, it also yields some results that are unrelated to, or perhaps even contrary to, everyday "ordinary" experience; it implies the existence of some rather bizarre, counterintuitive objects. Perhaps the most bizarre is the Banach - Tarski Paradox: At first glance, the Banach-Tarski result seems to contradict some of our intuition about physics -- e. If we assume that the ball has a uniform density, then the Banach-Tarski Paradox seems to say that we can disassemble a one-kilogram ball into pieces and rearrange them to get two one-kilogram balls. But actually, the contradiction can be explained away: Only a set with a defined volume can have a defined mass. A "volume" can be defined for many subsets of R3 spheres, cubes, cones, icosahedrons, etc. This leads beginners to expect that the notion of "volume" is applicable to every subset of R3. In particular, the pieces in the Banach-Tarski decomposition are sets whose volumes cannot be defined. More precisely, Lebesgue measure is defined on some subsets of R3, but it cannot be extended to all subsets of R3 in a fashion that preserves two of its most important properties: The pieces in the Banach-Tarski decomposition are not Lebesgue measurable. Thus, the Banach-Tarski Paradox gives as a corollary the fact that there exist sets that are not Lebesgue measurable. That corollary also has a much shorter proof not involving the Banach-Tarski Paradox which can be found in every introductory textbook on measure theory, but it too uses the Axiom of Choice. Here is a brief sketch of that shorter proof: Work in "the real numbers modulo 1" -- that is, the number system that you get if you cut the interval [0,1 out of the real line and loop it around into a circle, so that 0 and 1 are the same number. Like the way that 0 and 12 are the same on a circular clock. The union of these sets is all of T. There are a countable infinity of them. If the measure of C were zero, their sum would be zero. If the measure of C were positive, their sum would be infinite. Personally, I am not surprised to find the Axiom of Choice coming into play in a subject that is so inherently complicated as unmeasurable sets. I am much more surprised to find AC coming into play in this simpler and more concrete example: Define "small" to mean "finite. But it does not satisfy rule c, since the even numbers and the odd numbers are complements of each other, and neither of those sets is finite. Say that a set is "small" if the number 1 is not a member of that set. Say that a set is "small" if it contains at most one of the three numbers 1, 2, 3. That satisfies rules a and c. Does there exist a classification scheme satisfying all three rules? It turns out that such a classification scheme exists, but an example of such a classification scheme does not exist which makes it a bit hard to visualize! I mean that the proofs of existence are inherently nonconstructive -- i. But the proof of that fact is very deep, and it raises interesting philosophical questions: In what sense does that classification scheme "exist"? Technical details for experts: To prove the existence of such a classification scheme, just call "large" the members of some nonprincipal ultrafilter on the positive integers, and call their complements "small. An introduction to nonprincipal ultrafilters can be found in my book and in many other places in the literature. Let BP be the statement that "every subset of the reals has the Baire property. However, I would say that ZF is empirically consistent: In a century of study, mathematicians have not yet found any contradictions in ZF, despite the incentive that any mathematician finding such an important proof would instantly be promoted to full professor at any university in the world. My example with positive integers might appear to be simpler than the Banach-Tarski Paradox, but it does not really get us completely away from measure theory. A nonprincipal ultrafilter can be reformulated as a two-valued probability measure that is finitely additive but not countably additive.

DOWNLOAD PDF CONSEQUENCES OF THE AXIOM OF CHOICE

Chapter 2 : Consequences Of The Axiom Of Choice Book â€“ PDF Download

Consequences of the Axiom of Choice This book, Consequences of the Axiom of Choice, is a comprehensive listing of statements that have been proved in the last years using the axiom of choice. Each consequence, also referred to as a form of the axiom of choice, is assigned a number.

Qiaochu Yuan k Another way of loking at it is to realize that given a set A having volume 1 and a set B having volume 2, both sets have the same cardinality, i. That explains why if you can break A up into infinitely many points and B up into infinitely many points, you can turn one into the other. But the Banach-Tarski paradox accomplishes this with finitely many pieces. Yes, I realize that. I was merely pointing out that even without Banach-Tarski, our intuition about infinite sets can be deceiving. Consider a sphere with radius 1 and another with radius 2; both have different volumes but the same cardinalities. Banach-Tarski just takes that same non-intuitiveness to a higher level. And, if I remember correctly, you can do that with just four -five? This process, called existential instantiation, allows us to move from "the set is not empty" to "here is an element of the set". We cannot use existential instantiation infinitely many times. Remember that mathematics, formally, is always on its way to prove something. And to solve this we use the notion of a "choice function". Now we can apply existential instantiation to the set of choice functions, which we have proved to be non-empty using the axiom of choice, and obtain the wanted function. For this we need to talk about If we are given non-empty sets, then there is a way to choose an element from each set. But the consequences of the axiom of choice can be counterintuitive at first. But those people often mistake mathematical balls to actual physical balls or vice versa and a non-constructive mathematical process with what we can do by hand [or robot] in real life. The XKCD that you link is playing exactly on that. The character in the last panel has cut through the pumpkin several times, and suddenly there were two pumpkins. Just like in the Banach-Tarski paradox. This is not a complete account of the events, and there are more issues to care about. But I find that getting into them can be confusing, and initially it is a good idea to think about the problem as repeating instantiation.

DOWNLOAD PDF CONSEQUENCES OF THE AXIOM OF CHOICE

Chapter 3 : AMS eBooks: Mathematical Surveys and Monographs

In mathematics, the axiom of choice, or AC, is an axiom of set theory equivalent to the statement that the Cartesian product of a collection of non-empty sets is non-empty.

Among the axioms of ZF, perhaps the most attention has been devoted to 6 , the axiom of choice, which has a large number of equivalent formulations. Axioms for infinite and ordered sets]. The axiom of choice has the featureâ€”not shared by other axioms of set theoryâ€”that it asserts the existence of a set without ever specifying its elements or any definite way to select them. In general, S could have many choice functions. The axiom of choice merely asserts that it has at least one, without saying how to construct it. This nonconstructive feature has led to some controversy regarding the acceptability of the axiom. See also foundations of mathematics: The axiom of choice is not needed for finite sets since the process of choosing elements must come to an end eventually. For infinite sets , however, it would take an infinite amount of time to choose elements one by one. Thus, infinite sets for which there does not exist some definite selection rule require the axiom of choice or one of its equivalent formulations in order to proceed with the choice set. The English mathematician-philosopher Bertrand Russell gave the following succinct example of this distinction: Thus, without the axiom of choice, each sock would have to be chosen one by oneâ€”an eternal prospect. Nonetheless, the axiom of choice does have some counterintuitive consequences. The best-known of these is the Banach-Tarski paradox. This shows that for a solid sphere there exists in the sense that the axioms assert the existence of sets a decomposition into a finite number of pieces that can be reassembled to produce a sphere with twice the radius of the original sphere. Of course, the pieces involved are nonmeasurable; that is, one cannot meaningfully assign volumes to them. That is, the result of adding the axiom of choice to the other axioms ZFC remains consistent. Then in the American mathematician Paul Cohen completed the picture by showing, again under the assumption that ZF is consistent, that ZF does not yield a proof of the axiom of choice; that is, the axiom of choice is independent. In general, the mathematical community accepts the axiom of choice because of its utility and its agreement with intuition regarding sets. On the other hand, lingering unease with certain consequences such as well-ordering of the real numbers has led to the convention of explicitly stating when the axiom of choice is utilized, a condition not imposed on the other axioms of set theory.

DOWNLOAD PDF CONSEQUENCES OF THE AXIOM OF CHOICE

Chapter 4 : consequences of the axiom of choice | Download eBook PDF/EPUB

Consequences of the Axiom of Choice Project Homepage. The book Consequences of the Axiom of Choice by Paul Howard Send E-Mail to Paul Howard and Jean E. Rubin Send E- Mail to Jean Rubin is volume 59 in the series Mathematical Surveys and Monographs published by the American Mathematical Society in This book is a survey of research done during the last years on the axiom of choice and its consequences.

Each choice function on a collection X of nonempty sets is an element of the Cartesian product of the sets in X. This is not the most general situation of a Cartesian product of a family of sets, where a given set can occur more than once as a factor; however, one can focus on elements of such a product that select the same element every time a given set appears as factor, and such elements correspond to an element of the Cartesian product of all distinct sets in the family. The axiom of choice asserts the existence of such elements; it is therefore equivalent to: Given any family of nonempty sets, their Cartesian product is a nonempty set. AC â€" the Axiom of Choice. Variants[edit] There are many other equivalent statements of the axiom of choice. These are equivalent in the sense that, in the presence of other basic axioms of set theory, they imply the axiom of choice and are implied by it. One variation avoids the use of choice functions by, in effect, replacing each choice function with its range. Given any set X of pairwise disjoint non-empty sets, there exists at least one set C that contains exactly one element in common with each of the sets in X. Another equivalent axiom only considers collections X that are essentially powersets of other sets: For any set A, the power set of A with the empty set removed has a choice function. Authors who use this formulation often speak of the choice function on A, but be advised that this is a slightly different notion of choice function. Its domain is the powerset of A with the empty set removed , and so makes sense for any set A, whereas with the definition used elsewhere in this article, the domain of a choice function on a collection of sets is that collection, and so only makes sense for sets of sets. With this alternate notion of choice function, the axiom of choice can be compactly stated as Every set has a choice function. The negation of the axiom can thus be expressed as: There is a set A such that for all functions f on the set of non-empty subsets of A , there is a B such that f B does not lie in B. Restriction to finite sets[edit] The statement of the axiom of choice does not specify whether the collection of nonempty sets is finite or infinite, and thus implies that every finite collection of nonempty sets has a choice function. However, that particular case is a theorem of the Zermeloâ€"Fraenkel set theory without the axiom of choice ZF ; it is easily proved by mathematical induction. The axiom of choice can be seen as asserting the generalization of this property, already evident for finite collections, to arbitrary collections. Usage[edit] Until the late 19th century, the axiom of choice was often used implicitly, although it had not yet been formally stated. For example, after having established that the set X contains only non-empty sets, a mathematician might have said "let F s be one of the members of s for all s in X. Not every situation requires the axiom of choice. For finite sets X, the axiom of choice follows from the other axioms of set theory. In that case it is equivalent to saying that if we have several a finite number of boxes, each containing at least one item, then we can choose exactly one item from each box. Clearly we can do this: We start at the first box, choose an item; go to the second box, choose an item; and so on. The number of boxes is finite, so eventually our choice procedure comes to an end. The result is an explicit choice function: A formal proof for all finite sets would use the principle of mathematical induction to prove "for every natural number k, every family of k nonempty sets has a choice function. If the method is applied to an infinite sequence Xi: Examples[edit] The nature of the individual nonempty sets in the collection may make it possible to avoid the axiom of choice even for certain infinite collections. For example, suppose that each member of the collection X is a nonempty subset of the natural numbers. Every such subset has a smallest element, so to specify our choice function we can simply say that it maps each set to the least element of that set. This gives us a definite choice of an element from each set, and makes it unnecessary to apply the axiom of choice. The difficulty appears when there is no natural choice of elements from each set. If we cannot make explicit choices, how do we know that our set exists? For example, suppose that X is the set of all non-empty subsets of the real numbers. First we might try to proceed as if X were finite. If we try to choose an element from each set, then, because X is infinite, our choice procedure will never come to an end, and consequently, we shall never be able to produce a choice function for all of X. Next we might try specifying the least element from each set. But some subsets of the real numbers do not have least elements. For example, the open interval 0,1 does not have a least element: So this attempt also fails. Additionally, consider for instance the unit circle S, and the action on S by a group G consisting of all rational rotations. Here G is countable while S is uncountable. The set of those translates partitions the circle into a countable collection of disjoint sets, which are all pairwise congruent. Since X is not measurable for any rotation-invariant countably additive finite measure on S, finding an algorithm to select a point in each orbit requires the axiom of choice. See non-measurable set for more details. The reason that we are able to choose least elements from subsets of the natural numbers is the fact that the natural numbers are well-ordered: One might say, "Even though the usual ordering of the real numbers does not work, it may be possible to find a different ordering of the real numbers which is a well-ordering. Then our choice function can choose the least element of every set under our unusual ordering. Criticism and acceptance[edit] A proof requiring the axiom of choice may establish the existence of an object without explicitly defining the object in the language of set theory. For example, while the axiom of choice implies that there is a well-ordering of the real numbers, there are models of set theory with the axiom of choice in which no well-ordering of the reals is definable. Similarly, although a subset of the real numbers that is not Lebesgue measurable can be proved to exist using the axiom of choice, it is consistent that no such set is definable. This has been used as an argument against the use of the axiom of choice. Another argument against the axiom of choice is that it implies the existence of objects that may seem counterintuitive. The pieces in this decomposition, constructed using the axiom of choice, are non-measurable sets. Despite these seemingly paradoxical facts, most mathematicians accept the axiom of choice as a valid principle for proving new results in mathematics. The debate is interesting enough, however, that it is considered of note when a theorem in ZFC ZF plus AC is logically equivalent with just the ZF axioms to the axiom of choice, and mathematicians look for results that require the axiom of choice to be false, though this type of deduction is less common than the type which requires the axiom of choice to be true. It is possible to prove many theorems using neither the axiom of choice nor its negation; such statements will be true in any model of ZF, regardless of the truth or falsity of the axiom of choice in that particular model. The restriction to ZF renders any claim that relies on either the axiom of choice or its negation unprovable. For example, the Banachâ€”Tarski paradox is neither provable nor disprovable from ZF alone: Similarly, all the statements listed below which require choice or some weaker version thereof for their proof are unprovable in ZF, but since each is provable in ZF plus the axiom of choice, there are models of ZF in which each statement is true. Statements such as the Banachâ€”Tarski paradox can be rephrased as conditional statements, for example, "If AC holds, then the decomposition in the Banachâ€”Tarski paradox exists. In constructive mathematics[edit] As discussed above, in ZFC, the axiom of choice is able to provide " nonconstructive proofs " in which the existence of an object is proved although no explicit example is constructed. ZFC, however, is still formalized in classical logic. The axiom of choice has also been thoroughly studied in the context of constructive mathematics, where non-classical logic is employed. The status of the axiom of choice varies between different varieties of constructive mathematics. Thus the axiom of choice is not generally available in constructive set theory. A cause for this difference is that the axiom of choice in type theory does not have the extensionality properties that the axiom of choice in constructive set theory does. Although the axiom of countable choice in particular is commonly used in constructive mathematics, its use has also been questioned. In , Paul Cohen employed the technique of forcing , developed for this purpose, to show that: Together these results establish that the axiom of choice is logically independent of ZF. The assumption that ZF is consistent is harmless because adding another axiom to an already inconsistent system cannot make the situation worse. Because of independence, the decision whether to use the axiom of choice or its negation in a proof cannot be made by appeal to other axioms of set theory. The decision must be made on other grounds. One argument given in favor of using the axiom of choice is that it is convenient to use it because it allows one to prove some simplifying propositions that otherwise could not be proved. Many theorems which are provable using choice are of an elegant general character: Without the axiom of choice, these theorems may not hold for mathematical objects of large cardinality. The proof of the independence result also shows that a wide class of mathematical statements, including all statements that can be phrased in the language of Peano arithmetic , are provable in ZF if and only if they are provable in ZFC. When one attempts to solve problems in this class, it makes no difference whether ZF or ZFC is employed if the only question is the existence of a proof. The axiom of choice is not the only significant statement which is independent of ZF. Stronger axioms[edit] The axiom of constructibility and the generalized continuum hypothesis each imply the axiom of choice and so are strictly stronger than it. The axiom of global choice follows from the axiom of limitation of size. In fact, Zermelo initially introduced the axiom of choice in order to formalize his proof of the well-ordering theorem. Every set can be well-ordered. Consequently, every cardinal has an initial ordinal. If two sets are given, then either they have the same cardinality, or one has a smaller cardinality than the other. Given two non-empty sets, one has a surjection to the other. The Cartesian product of any family of nonempty sets is nonempty. Colloquially, the sum of a sequence of cardinals is strictly less than the product of a sequence of larger cardinals. The reason for the term "colloquially" is that the sum or product of a "sequence" of cardinals cannot be defined without some aspect of the axiom of choice. Every non-empty partially ordered set in which every chain i. In any partially ordered set, every totally ordered subset is contained in a maximal totally ordered subset. The restricted principle "Every partially ordered set has a maximal totally ordered subset" is also equivalent to AC over ZF. Every non-empty collection of finite character has a maximal element with respect to inclusion. Every partially ordered set has a maximal antichain.

Chapter 5 : countable choice in nLab

This book, ""Consequences of the Axiom of Choice"", is a comprehensive listing of statements that have been proved in the last years using the axiom of choice. Each consequence, also referred to as a form of the axiom of choice, is assigned a number.

You have printed 0 times in the last 24 hours. Your print count will reset on at. You may print 0 more time s before then. You may print a maximum of 0 pages at a time. Number of pages to print: Printing will start on the current page. Firefox users may need to click "Back" when printing completes. Print 0 Pages Consequences of the Axiom of Choice Page 1 10 of GO Background In this book we have attempted to collect most of the consequences of the ax- iom of choice that have been discovered during the past 90 years. Our goal is to present these statements and the relationships between them in an easily accessible format. Since , when Zermelo introduced the axiom of choice, hundreds of its consequences have appeared in articles by almost as many authors. These articles have explored the complicated pattern of implications between the various conse- quences. With the introduction of independence proofs by Fraenkel, Mostowski, Specker and Cohen, the pattern of relationships became still more intricate. The results in this area are scattered over a wide spectrum of journals and books. As a result a particular theorem may be proved and reproved by several different au- thors. For example, Shannon [] asked whether or not the axiom of choice for countable families of countable sets implies the countable union theorem. Shortly afterward Norbert Brunner, in his review of Howard [], pointed out that the result had already been obtained by Feigner in []. This book was begun with the primary purpose of preventing such occurrences. We hope it will also serve as a reference for non-set theorist who need to know the strength of some particular statement relative to the axiom of choice and its consequences. However, our task is complicated by the fact that the statements we consider are not, in general, equivalent. We are concerned with sentences in the language of set theory which can be proved using the axiom of choice but which are not theorems of set theory with that axiom omitted with the exception of form 0 described below. We call these sentences forms of the axiom of choice. The forms are numbered starting with form 0. The remaining forms are numbered in the order we encountered them as we reviewed the literature. We consider two versions of set theory. The first is Zermelo-Fraenkel set theory which we denote by ZF. For our purposes any one of the axiomatizations of the theory ZF will be satisfactory. To be specific we shall use the one given in Jech [b], page In addition the axioms of extensionality Al in Jech [b] and foundation or regularity A8 in Jech [b] are mod- ified to allow the existence of atoms or urelements. Please report unauthorized use to cust-serv ams. Consequences of the Axiom of Choice resources Help.

Chapter 6 : Axiom of choice - Wikipedia

There are certain axioms contradicting the axiom of choice, however, that have been studied a lot, like the axiom of determinacy (AD) or the measurability of all sets of reals. Views Ryan Reich, Ph.D Mathematics, Harvard University.

This is now usually stated in terms of choice functions: A different choice function is obtained by assigning to each pair its greatest element. Any collection of nonempty sets has a choice function. AC1 can be reformulated in terms of indexed or variable sets. AC1 is then equivalent to the assertion AC2: Any indexed collection of sets has a choice function. Informally speaking, AC2 amounts to the assertion that a variable set with an element at each stage has a variable element. AC1 can also be reformulated in terms of relations, viz. Finally AC3 is easily shown to be equivalent in the usual set theories to: Any surjective function has a right inverse. In a paper Zermelo introduced a modified form of AC. Any collection of mutually disjoint nonempty sets has a transversal. It is to be noted that AC1 and CAC for finite collections of sets are both provable by induction in the usual set theories. But in the case of an infinite collection, even when each of its members is finite, the question of the existence of a choice function or a transversal is problematic[4]. For example, as already mentioned, it is easy to come up with a choice function for the collection of pairs of real numbers simply choose the smaller element of each pair. But it is by no means obvious how to produce a choice function for the collection of pairs of arbitrary sets of real numbers. The chief objection raised was to what some saw as its highly non-constructive, even idealist, character: In the first of these, as remarked above, he reformulated AC in terms of transversals; in the second a he made explicit the further assumptions needed to carry through his proof of the well-ordering theorem. These assumptions constituted the first explicit presentation of an axiom system for set theory. As the debate concerning the Axiom of Choice rumbled on, it became apparent that the proofs of a number of significant mathematical theorems made essential use of it, thereby leading many mathematicians to treat it as an indispensable tool of their trade. Hilbert, for example, came to regard AC as an essential principle of mathematics[5] and employed it in his defence of classical mathematical reasoning against the attacks of the intuitionists. Although the usefulness of AC quickly become clear, doubts about its soundness remained. These doubts were reinforced by the fact that it had certain strikingly counterintuitive consequences. Here is a brief chronology of AC: Here by an atom is meant a pure individual, that is, an entity having no members and yet distinct from the empty set so a fortiori an atom cannot be a set. This had to wait until when Paul Cohen showed that it is consistent with the standard axioms of set theory which preclude the existence of atoms to assume that a countable collection of pairs of sets of real numbers fails to have a choice function. He introduced a new hierarchy of setsâ€"the constructible hierarchyâ€"by analogy with the cumulative type hierarchy. The relative consistency of AC with ZF follows. Broadly speaking, these propositions assert that certain conditions are sufficient to ensure that a partially ordered set contains at least one maximal element, that is, an element such that, with respect to the given partial ordering, no element strictly exceeds it. To state it, we need a few definitions. Every nonempty inductive partially ordered set has a maximal element. Here is an informal argument. This may in turn be formulated in a dual form. Call a family of sets strongly reductive if it is closed under intersections of nests. Then any nonempty strongly reductive family of sets has a minimal element, that is, a member properly including no member of the family. Here is a brief chronology of maximal principles. It seems to have been Artin who first recognized that ZL would yield AC, so that the two are equivalent over the remaining axioms of set theory. The Axiom of Choice has numerous applications in mathematics, a number of which have proved to be formally equivalent to it[13]. Historically the most important application was the first, namely: The Well-Ordering Theorem Zermelo , Every set can be well-ordered. After Zermelo published his proof of the well-ordering theorem from AC, it was quickly seen that the two are equivalent. The product of any set of non-zero cardinal numbers is non-zero. Early applications of AC include: Every infinite set has a denumerable subset. This principle, again weaker than AC, cannot be proved without it in the context of the remaining axioms of set theory. Every infinite cardinal number is equal to its square. This was proved equivalent to AC in Tarski Every vector space has a basis initiated by Hamel This was proved equivalent to AC in Blass

Every field has an algebraic closure Steinitz This assertion is weaker than AC, indeed is a consequence of the weaker compactness theorem for first-order logic see below. There is a Lebesgue nonmeasurable set of real numbers Vitali Solovay established its independence of the remaining axioms of set theory. This principle, although much weaker than AC, cannot be proved without it in the context of the remaining axioms of set theory. Mathematical equivalents of AC include: This was proved equivalent to AC in Kelley This was proved equivalent to AC by Tarski. This was proved equivalent to AC in Bell and Fremlin a. Every distributive lattice has a maximal ideal. This was proved equivalent to AC in Klimovsky , and for lattices of sets in Bell and Fremlin Every commutative ring with identity has a maximal ideal. This was proved equivalent to AC by Hodges There are a number of mathematical consequences of AC which are known to be weaker[14] than it, in particular: This was shown to be weaker than AC in Halpern and Levy If the cardinality of the model is specified in the right way, the assertion becomes equivalent to AC. The question of the equivalence of this with AC is one of the few remaining interesting open questions in this area; while it clearly implies BPI, it was proved independent of BPI in Bell Many of these theorems are discussed in Bell and Machover The scheme of sentences AC1L: Here predicates are playing the role of sets. Up to now we have tacitly assumed our background logic to be the usual classical logic. But the true depth of the connection between AC and logic emerges only when intuitionistic or constructive logic is brought into the picture. The fact that the Axiom of Choice implies Excluded Middle seems at first sight to be at variance with the fact that the former is often taken as a valid principle in systems of constructive mathematics governed by intuitionistic logic, e. To resolve the difficulty, we note that in deriving Excluded Middle from ACL1 essential use was made of the principles of Predicative Comprehension and Extensionality of Functions[18]. It follows that, in systems of constructive mathematics affirming AC but not Excluded Middle either the principle of Predicative Comprehension or the principle of Extensionality of Functions must fail. While the principle of Predicative Comprehension can be given a constructive justification, no such justification can be provided for the principle of Extensionality of Functions. In intuitionistic set theory that is, set theory based on intuitionistic as opposed to classical logicâ€"we shall abbreviate this as IST and in topos theory the principles of Predicative Comprehension and Extensionality of Functions both appropriately construed hold and so there AC implies Excluded Middle. His proof employed essentially different ideas from the proof presented above; in particular, it makes no use of extensionality principles but instead employs the idea of the quotient of an object or set by an equivalence relation. Now we show that, if AC4 holds, then any subset of a set has an indicator, and hence is detachable. It can be shown Bell that each of a number of intuitionistically invalid logical principles, including the law of excluded middle, is equivalent in intuitionistic set theory to a suitably weakened version of the axiom of choice. Accordingly these logical principles may be viewed as choice principles. Here are the logical principles at issue: All of these schemes follow, of course, from the full law of excluded middle, that is SLEM for arbitrary formulas. Each of the logical principles tabulated above is equivalent in IST to a choice principle. These results show just how deeply choice principles interact with logic, when the background logic is assumed to be intuitionistic. In a classical setting where the Law of Excluded Middle is assumed these connections are obliterated. Readers interested in the topic of the axiom of choice and type theory may consult the following supplementary document: Brouwer Centenary Symposium, Amsterdam: Notes on Constructive Set Theory. Toposes and Local Set Theories: Boolean-valued Models and Independence Proofs, Oxford: The Axiom of Choice, London: A Course in Mathematical Logic. Elements de Mathematique, Livre I: Theorie des Ensembles, Paris: National Academy of Sciemces, Foundations of Set Theory, Amsterdam: National Academy of Sciences, The Journal of the Bertrand Russell Archives, 32 1:

DOWNLOAD PDF CONSEQUENCES OF THE AXIOM OF CHOICE

Chapter 7 : Axiom of choice | set theory | www.nxgvision.com

The principle of set theory known as the Axiom of Choice has been hailed as "probably the most interesting and, in spite of its late appearance, the most discussed axiom of mathematics, second only to Euclid's axiom of parallels which was introduced more than two thousand years ago" (Fraenkel, Bar-Hillel & Levy , Â§II.4).

This book, Consequences of the Axiom of Choice, is a comprehensive listing of statements that have been proved in the last years using the axiom of choice. Each consequence, also referred to as a form of the axiom of choice, is assigned a number. Part I is a listing of the forms by number. In this part each form is given together with a listing of all statements known to be equivalent to it equivalent in set theory without the axiom of choice. In Part II the forms are arranged by topic. In Part III we describe the models of set theory which are used to show non-implications between forms. Part IV, the notes section, contains definitions, summaries of important sub-areas and proofs that are not readily available elsewhere. Part V gives references for the relationships between forms and Part VI is the bibliography. Part VII is contained on the floppy disk which is enclosed in the book. It contains a table with form numbers as row and column headings. Software for easily extracting information from the table is also provided. Tables 1 and 2 are accessible on the PC-compatible software included with the book. In addition, the program maketex. Cambridge University Press Format Available: Following the success of Logic for Mathematicians, Dr Hamilton has written a text for mathematicians and students of mathematics that contains a description and discussion of the fundamental conceptual and formal apparatus upon which modern pure mathematics relies. He emphasises the intuitive basis of mathematics; the basic notions are numbers and sets and they are considered both informally and formally. The role of axiom systems is part of the discussion but their limitations are pointed out. Formal set theory has its place in the book but Dr Hamilton recognises that this is a part of mathematics and not the basis on which it rests. Throughout, the abstract ideas are liberally illustrated by examples so this account should be well-suited, both specifically as a course text and, more broadly, as background reading. The reader is presumed to have some mathematical experience but no knowledge of mathematical logic is required. The reader should derive from this volume a maximum of understanding of the theorems of set theory and of their basic importance in the study of mathematics.

Chapter 8 : Consequences of the Axiom of Choice - Paul Howard, Jean E. Rubin - Google Books

For example, the known equivalences between definitions of finite sets are stated, and the known truth or falsity of choice axioms and numerous choice-dependent theorems in algebra, graph theory, topology, logic and cardinal numbers.

Chapter 9 : Karl-Heinz Diener, On Îº-hereditary Sets and Consequences of the Axiom of Choice - PhilPape

The axiom of choice is an axiom in set theory with wide-reaching and sometimes counterintuitive consequences. It states that for any collection of sets, one can construct a new set containing an element from each set in the original collection.