# DOWNLOAD PDF FOUNDATIONS OF MATHEMATICAL LOGIC

## Chapter 1 : Logic and Mathematics

*This Dover book, Foundations of Mathematical Logic, by Haskell Brooks Curry, originally published in , summarizes pretty much every approach to logic up to that time. Although there is a chapter at the end on modal logic, it's mostly concerned with the kinds of logics which are directly applicable to real-world mathematics.*

Ancient Greek mathematics While the practice of mathematics had previously developed in other civilizations, special interest in its theoretical and foundational aspects was clearly evident in the work of the Ancient Greeks. Early Greek philosophers disputed as to which is more basic, arithmetic or geometry. The Pythagorean school of mathematics originally insisted that only natural and rational numbers exist. Aristotle took a majority of his examples for this from arithmetic and from geometry. Euclid justifies each proposition by a demonstration in the form of chains of syllogisms though they do not always conform strictly to Aristotelian templates. Platonism as a traditional philosophy of mathematics[ edit ] This section needs additional citations for verification. Please help improve this article by adding citations to reliable sources. Unsourced material may be challenged and removed. October Further information: Platonism mathematics Starting from the end of the 19th century, a Platonist view of mathematics became common among practicing mathematicians. The concepts or, as Platonists would have it, the objects of mathematics are abstract and remote from everyday perceptual experience: Their existence and nature present special philosophical challenges: How do mathematical objects differ from their concrete representation? Are they located in their representation, or in our minds, or somewhere else? How can we know them? The ancient Greek philosophers took such questions very seriously. Indeed, many of their general philosophical discussions were carried on with extensive reference to geometry and arithmetic. He believed that the truths about these objects also exist independently of the human mind, but is discovered by humans. In this way Plato indicated his high opinion of geometry. He regarded geometry as "the first essential in the training of philosophers", because of its abstract character. This philosophy of Platonist mathematical realism is shared by many mathematicians. It can be argued that Platonism somehow comes as a necessary assumption underlying any mathematical work. Not our axioms, but the very real world of mathematical objects forms the foundation. Aristotle dissected and rejected this view in his Metaphysics. These questions provide much fuel for philosophical analysis and debate. The Middle Ages saw a dispute over the ontological status of the universals platonic Ideas: Realism asserted their existence independently of perception; conceptualism asserted their existence within the mind only; nominalism denied either, only seeing universals as names of collections of individual objects following older speculations that they are words, "logoi". Isaac Newton â€” in England and Leibniz â€” in Germany independently developed the infinitesimal calculus based on heuristic methods greatly efficient, but direly lacking rigorous justifications. Leibniz even went on to explicitly describe infinitesimals as actual infinitely small numbers close to zero. Leibniz also worked on formal logic but most of his writings on it remained unpublished until  The Protestant philosopher George Berkeley â€” , in his campaign against the religious implications of Newtonian mechanics, wrote a pamphlet on the lack of rational justifications of infinitesimal calculus: May we not call them the ghosts of departed quantities? Concerns about logical gaps and inconsistencies in different fields led to the development of axiomatic systems. But he did not formalize his notion of convergence. Mathematicians such as Karl Weierstrass â€” discovered pathological functions such as continuous, nowhere-differentiable functions. Previous conceptions of a function as a rule for computation, or a smooth graph, were no longer adequate. Weierstrass began to advocate the arithmetization of analysis , to axiomatize analysis using properties of the natural numbers. In , Dedekind proposed a definition of the real numbers as cuts of rational numbers. This reduction of real numbers and continuous functions in terms of rational numbers, and thus of natural numbers, was later integrated by Cantor in his set theory, and axiomatized in terms of second order arithmetic by Hilbert and Bernays. History of group theory For the first time, the limits of mathematics were explored. With these concepts, Pierre Wantzel proved that straightedge and compass alone cannot trisect an arbitrary angle nor double a cube. Mathematicians had attempted to solve all of these problems in vain since the time of the ancient Greeks. Geometry was no more limited to three dimensions. These concepts did not generalize numbers but combined notions of functions and sets which were not yet formalized, breaking away from familiar mathematical objects. It was proved consistent by defining point to mean a pair of antipodal points on a fixed sphere and line to mean a great circle on the sphere. At that time, the main method for proving the consistency of a set of axioms was to provide a model for it. Projective geometry[ edit ] One of the traps in a deductive system is circular reasoning , a problem that seemed to befall projective geometry until it was resolved by Karl von Staudt. As explained by Russian historians: Indeed the basic concept that is applied in the synthetic presentation of projective geometry, the cross-ratio of four points of a line, was introduced through consideration of the lengths of intervals. The purely geometric approach of von Staudt was based on the complete quadrilateral to express the relation of projective harmonic conjugates. Then he created a means of expressing the familiar numeric properties with his Algebra of Throws. Stillwell writes on page  The algebra of throws is commonly seen as a feature of cross-ratios since students ordinarily rely upon numbers without worry about their basis. However, cross-ratio calculations use metric features of geometry, features not admitted by purists. For instance, in Coxeter wrote Introduction to Geometry without mention of cross-ratio. Boolean algebra and logic[ edit ] Attempts of formal treatment of mathematics had started with Leibniz and Lambert â€" , and continued with works by algebraists such as George Peacock â€" Systematic mathematical treatments of logic came with the British mathematician George Boole who devised an algebra that soon evolved into what is now called Boolean algebra , in which the only numbers were 0 and 1 and logical combinations conjunction, disjunction, implication and negation are operations similar to the addition and multiplication of integers. Additionally, De Morgan published his laws in  Logic thus became a branch of mathematics. Boolean algebra is the starting point of mathematical logic and has important applications in computer science. Charles Sanders Peirce built upon the work of Boole to develop a logical system for relations and quantifiers , which he published in several papers from to  The German mathematician Gottlob Frege â€" presented an independent development of logic with quantifiers in his Begriffsschrift formula language published in , a work generally considered as marking a turning point in the history of logic. He then showed in Grundgesetze der Arithmetik Basic Laws of Arithmetic how arithmetic could be formalised in his new logic. This work summarized and extended the work of Boole, De Morgan, and Peirce, and was a comprehensive reference to symbolic logic as it was understood at the end of the 19th century. Peano arithmetic The formalization of arithmetic the theory of natural numbers as an axiomatic theory started with Peirce in and continued with Richard Dedekind and Giuseppe Peano in  This was still a second-order axiomatization expressing induction in terms of arbitrary subsets, thus with an implicit use of set theory as concerns for expressing theories in first-order logic were not yet understood. The name "paradox" should not be confused with contradiction. But a paradox may be either a surprising but true result in a given formal theory, or an informal argument leading to a contradiction, so that a candidate theory, if it is to be formalized, must disallow at least one of its steps; in this case the problem is to find a satisfying theory without contradiction. Both meanings may apply if the formalized version of the argument forms the proof of a surprising truth. Various schools of thought opposed each other. The main opponent was the intuitionist school, led by L. Brouwer , which resolutely discarded formalism as a meaningless game with symbols van Dalen,  The fight was acrimonious. In Hilbert succeeded in having Brouwer, whom he considered a threat to mathematics, removed from the editorial board of Mathematische Annalen , the leading mathematical journal of the time.

## Chapter 2 : PDF Mathematical Logic Dover Books On Mathematics Free Download | Download PDF Journ

*Foundations of mathematics is the study of the philosophical and logical and/or algorithmic basis of mathematics, or, in a broader sense, the mathematical investigation of what underlies the philosophical theories concerning the nature of mathematics.*

Each area has a distinct focus, although many techniques and results are shared among multiple areas. The borderlines amongst these fields, and the lines separating mathematical logic and other fields of mathematics, are not always sharp. The method of forcing is employed in set theory, model theory, and recursion theory, as well as in the study of intuitionistic mathematics. The mathematical field of category theory uses many formal axiomatic methods, and includes the study of categorical logic , but category theory is not ordinarily considered a subfield of mathematical logic. Because of its applicability in diverse fields of mathematics, mathematicians including Saunders Mac Lane have proposed category theory as a foundational system for mathematics, independent of set theory. These foundations use toposes , which resemble generalized models of set theory that may employ classical or nonclassical logic. History[ edit ] Mathematical logic emerged in the midth century as a subfield of mathematics, reflecting the confluence of two traditions: The first half of the 20th century saw an explosion of fundamental results, accompanied by vigorous debate over the foundations of mathematics. History of logic Theories of logic were developed in many cultures in history, including China , India , Greece and the Islamic world. In 18th-century Europe, attempts to treat the operations of formal logic in a symbolic or algebraic way had been made by philosophical mathematicians including Leibniz and Lambert , but their labors remained isolated and little known. Charles Sanders Peirce built upon the work of Boole to develop a logical system for relations and quantifiers, which he published in several papers from to Gottlob Frege presented an independent development of logic with quantifiers in his Begriffsschrift , published in , a work generally considered as marking a turning point in the history of logic. The two-dimensional notation Frege developed was never widely adopted and is unused in contemporary texts. This work summarized and extended the work of Boole, De Morgan, and Peirce, and was a comprehensive reference to symbolic logic as it was understood at the end of the 19th century. Foundational theories[ edit ] Concerns that mathematics had not been built on a proper foundation led to the development of axiomatic systems for fundamental areas of mathematics such as arithmetic, analysis, and geometry. In logic, the term arithmetic refers to the theory of the natural numbers. Around the same time Richard Dedekind showed that the natural numbers are uniquely characterized by their induction properties. In addition to the independence of the parallel postulate , established by Nikolai Lobachevsky in Lobachevsky , mathematicians discovered that certain theorems taken for granted by Euclid were not in fact provable from his axioms. Among these is the theorem that a line contains at least two points, or that circles of the same radius whose centers are separated by that radius must intersect. Hilbert developed a complete set of axioms for geometry , building on previous work by Pasch The success in axiomatizing geometry motivated Hilbert to seek complete axiomatizations of other areas of mathematics, such as the natural numbers and the real line. This would prove to be a major area of research in the first half of the 20th century. The 19th century saw great advances in the theory of real analysis , including theories of convergence of functions and Fourier series. Mathematicians such as Karl Weierstrass began to construct functions that stretched intuition, such as nowhere-differentiable continuous functions. Previous conceptions of a function as a rule for computation, or a smooth graph, were no longer adequate. Weierstrass began to advocate the arithmetization of analysis , which sought to axiomatize analysis using properties of the natural numbers. In , Dedekind proposed a definition of the real numbers in terms of Dedekind cuts of rational numbers Dedekind , a definition still employed in contemporary texts. Georg Cantor developed the fundamental concepts of infinite set theory. His early results developed the theory of cardinality and proved that the reals and the natural numbers have different cardinalities Cantor Over the next twenty years, Cantor developed a theory of transfinite numbers in a series of publications. Cantor believed that every set could be well-ordered , but was unable to produce a proof for this result, leaving it as an open problem in Katz , p. The discovery of paradoxes in informal set theory caused some to wonder whether mathematics itself is inconsistent, and to look for proofs of consistency. In , Hilbert posed a famous list of 23 problems for the next century. The first two of these were to resolve the continuum hypothesis and prove the consistency of elementary arithmetic, respectively; the tenth was to produce a method that could decide whether a multivariate polynomial equation over the integers has a solution. This problem asked for a procedure that would decide, given a formalized mathematical statement, whether the statement is true or false. Set theory and paradoxes[ edit ] Ernst Zermelo gave a proof that every set could be well-ordered, a result Georg Cantor had been unable to obtain. To achieve the proof, Zermelo introduced the axiom of choice , which drew heated debate and research among mathematicians and the pioneers of set theory. The immediate criticism of the method led Zermelo to publish a second exposition of his result, directly addressing criticisms of his proof Zermelo a. This paper led to the general acceptance of the axiom of choice in the mathematics community. Skepticism about the axiom of choice was reinforced by recently discovered paradoxes in naive set theory. Cesare Burali-Forti was the first to state a paradox: Zermelo b provided the first set of axioms for set theory. These axioms, together with the additional axiom of replacement proposed by Abraham Fraenkel , are now called Zermeloâ€"Fraenkel set theory ZF. This seminal work developed the theory of functions and cardinality in a completely formal framework of type theory , which Russell and Whitehead developed in an effort to avoid the paradoxes. Later work by Paul Cohen showed that the addition of urelements is not needed, and the axiom of choice is unprovable in ZF. Skolem realized that this theorem would apply to first-order formalizations of set theory, and that it implies any such formalization has a countable model. These results helped establish first-order logic as the dominant logic used by mathematicians. It showed the impossibility of providing a consistency proof of arithmetic within any formal theory of arithmetic. Hilbert, however, did not acknowledge the importance of the incompleteness theorem for some time. This leaves open the possibility of consistency proofs that cannot be formalized within the system they consider. Gentzen proved the consistency of arithmetic using a finitistic system together with a principle of transfinite induction. Beginnings of the other branches[ edit ] Alfred Tarski developed the basics of model theory. Beginning in , a group of prominent mathematicians collaborated under the pseudonym Nicolas Bourbaki to publish a series of encyclopedic mathematics texts. These texts, written in an austere and axiomatic style, emphasized rigorous presentation and set-theoretic foundations. Terminology coined by these texts, such as the words bijection, injection, and surjection , and the set-theoretic foundations the texts employed, were widely adopted throughout mathematics. Kleene introduced the concepts of relative computability, foreshadowed by Turing , and the arithmetical hierarchy. Kleene later generalized recursion theory to higher-order functionals. Kleene and Kreisel studied formal versions of intuitionistic mathematics, particularly in the context of proof theory. Formal logical systems [ edit ] At its core, mathematical logic deals with mathematical concepts expressed using formal logical systems. These systems, though they differ in many details, share the common property of considering only expressions in a fixed formal language. The systems of propositional logic and first-order logic are the most widely studied today, because of their applicability to foundations of mathematics and because of their desirable proof-theoretic properties. First-order logic First-order logic is a particular formal system of logic. Its syntax involves only finite expressions as well-formed formulas, while its semantics are characterized by the limitation of all quantifiers to a fixed domain of discourse. Early results from formal logic established limitations of first-order logic. This shows that it is impossible for a set of first-order axioms to characterize the natural numbers, the real numbers, or any other infinite structure up to isomorphism. As the goal of early foundational studies was to produce axiomatic theories for all parts of mathematics, this limitation was particularly stark. It shows that if a particular sentence is true in every model that satisfies a particular set of axioms, then there must be a finite deduction of the sentence from the axioms. It says that a set of sentences has a model if and only if every finite subset has a model, or in other words that an inconsistent set of formulas must have a finite inconsistent subset. The completeness and compactness theorems allow for sophisticated analysis of logical consequence in first-order logic and the development of model theory , and they are a key reason for the prominence of first-order logic in mathematics. The first incompleteness theorem states that for any consistent, effectively given defined below logical system that is capable of interpreting arithmetic, there exists a statement that is true in the sense that it holds for the natural numbers but not provable within that logical system and which indeed may fail in some non-standard models of arithmetic which may be consistent with the logical system. Here a logical system is said to be effectively given if it is possible to decide, given any formula in the language of the system, whether the formula is an axiom, and one which can express the Peano axioms is called "sufficiently strong. Other classical logics[ edit ] Many logics besides first-order logic are studied. These include infinitary logics , which allow for formulas to provide an infinite amount of information, and higher-order logics , which include a portion of set theory directly in their semantics. The most well studied infinitary logic is L.

## Chapter 3 : Foundations of mathematics - Wikipedia

*This book is a thoroughly documented and comprehensive account of the constructive theory of the first-order predicate calculus. This is a calculus that is central to modern mathematical logic and important for mathematicians, philosophers, and scientists whose work impinges upon logic. Professor.*

This is similar to the Aristotelean syllogism, but it is of wider applicability, because the premises and the conclusion can be more complex. As an example, the 19th century logician Augustus DeMorgan noted 9 that the inference all horses are animals, therefore, the head of a horse is the head of an animal is beyond the reach of Aristotelean logic. The completeness theorem Formulas of the predicate calculus can be exceedingly complicated. How then can we distinguish the formulas that are logically valid from the formulas that are not logically valid? It turns out that there is an algorithm 10 for recognizing logically valid formulas. We shall now sketch this algorithm. In order to recognize that a formula is logically valid, it suffices to construct what is known as a proof tree for , or equivalently a refutation tree for. This is a tree which carries at the root. Each node of the tree carries a formula. The growth of the tree is guided by the meaning of the logical operators appearing in. New nodes are added to the tree depending on what nodes have already appeared. For example, if a node carrying has appeared, we create two new nodes carrying and respectively. The thought behind these new nodes is that the only way for to be the case is if at least one of or is the case. Similarly, if a node carrying has already appeared, we create a new node carrying , where is the result of substituting a new constant for the variable. The idea here is that the only way for the universal statement to be false is if is false for some particular. Since is a new constant, is a formula which may be considered as the most general false instance of. Corresponding to each of the seven logical operators, there are prescribed procedures for adding new nodes to the tree. We apply these procedures repeatedly until they cannot be applied any more. If explicit contradictions 11 are discovered along each and every branch of the tree, then we have a refutation tree for. Thus is seen to be logically impossible. In other words, is logically valid. The adequacy of proof trees for recognizing logically valid formulas is a major insight of 20th century logic. On the other hand, the class of logically valid formulas is known to be extremely complicated. Indeed, this class is undecidable: In this sense, the concept of logical validity is too general and too intractable to be analyzed thoroughly. There will never be a predicate calculus analog of the pons asinorum. Formal theories The predicate calculus is a very general and flexible framework for reasoning. By choosing appropriate predicates, one can reason about any subject whatsoever. These considerations lead to the notion of a formal theory. In order to specify a formal theory, one first chooses a small collection of predicates which are regarded as basic for a given field of study. These predicates are the primitives of the theory. They delimit the scope of the theory. Other predicates must be defined in terms of the primitives. Using them, one writes down certain formulas which are regarded as basic or self-evident within the given field of study. These formulas are the axioms of the theory. It is crucial to make all of our underlying assumptions explicit as axioms. Once this has been done, a theorem is any formula which is a logical consequence of the axioms. A formal theory is this structure of primitives, axioms, and theorems. As a frivolous example, we could envision a theory of cars, trucks, and drivers. The defining axioms for and would be and , respectively. In this fashion, we could attempt to codify all available knowledge about vehicles and drivers. More seriously, one could try to write down formal theories corresponding to various scientific disciplines, such as mechanics or statistics or law. In this way one could hope to analyze the logical structure of the respective disciplines. The process of codifying a scientific discipline by means of primitives and axioms in the predicate calculus is known as formalization. The key issue here is the choice of primitives and axioms. They cannot be chosen arbitrarily. The scientist who chooses them must exercise a certain aesthetic touch. They must be small in number; they must be basic and self-evident; and they must account for the largest possible number of other concepts and facts. To date, this kind of formal theory-building has been convincingly carried out in only a few cases. A survey is in Tarski [ 21 ]. The most notable successes have been in mathematics. Foundations of mathematics Mathematics is the science of quantity. Traditionally there were two branches of mathematics, arithmetic and geometry, dealing with two kinds of quantities: Modern mathematics is richer and deals with a wider variety of objects, but arithmetic and geometry are still of central importance. Foundations of mathematics is the study of the most basic concepts and logical structure of mathematics, with an eye to the unity of human knowledge. Among the most basic mathematical concepts are: The reader may reasonably ask why mathematics appears at all in this volume. There are three reasons for discussing mathematics in a volume on general philosophy: Mathematics has always played a special role in scientific thought. The abstract nature of mathematical objects presents philosophical challenges that are unusual and unique. Foundations of mathematics is a subject that has always exhibited an unusually high level of technical sophistication. For this reason, many thinkers have conjectured that foundations of mathematics can serve as a model or pattern for foundations of other sciences. The philosophy of mathematics has served as a highly articulated test-bed where mathematicians and philosophers alike can explore how various general philosophical doctrines play out 13 in a specific scientific context. The purpose of this section is to indicate the role of logic in the foundations of mathematics. We begin with a few remarks on the geometry of Euclid. We then describe some modern formal theories for mathematics. Let no one who is ignorant of geometry enter here. In this way Plato indicated his high opinion of geometry. See also Plato [ 17 , Republic, B]. In the Posterior Analytics [ 13 ], Aristotle laid down the basics of the scientific method. Euclid begins with 21 definitions, five postulates, and five common notions. After that, the rest of the Elements are an elaborate deductive structure consisting of hundreds of propositions. Each proposition is justified by its own demonstration. The demonstrations are in the form of chains of syllogisms. In each syllogism, the premises are identified as coming from among the definitions, postulates, common notions, and previously demonstrated propositions. It is true that the syllogisms of Euclid do not always conform strictly to Aristotelean templates. The logic of Aristotle and the geometry of Euclid are universally recognized as towering scientific achievements of ancient Greece. Formal theories for mathematics A formal theory for geometry With the advent of calculus in the 17th and 18th centuries, mathematics developed very rapidly and with little attention to logical foundations. But the prolific Enlightenment mathematicians such as Leonhard Euler showed almost no interest in trying to place calculus on a similarly firm foundation. Only in the last half of the 19th century did scientists begin to deal with this foundational problem in earnest. The resulting crisis had far-reaching consequences. Geometers such as Moritz Pasch discovered what they regarded as gaps or inaccuracies in the Elements. Great mathematicians such as David Hilbert entered the fray. An outcome of all this foundational activity was a thorough reworking of geometry, this time as a collection of formal theories within the predicate calculus. Decisive insights were obtained by Alfred Tarski.

# DOWNLOAD PDF FOUNDATIONS OF MATHEMATICAL LOGIC

## Chapter 4 : Foundations of Mathematical Logic by Haskell B. Curry

*Written by a pioneer of mathematical logic, this comprehensive graduate-level text explores the constructive theory of first-order predicate calculus. It covers formal methods â€" including algorithms and epitheory â€" and offers a brief treatment of Markov's approach to algorithms. It also.*

However, the first to exhibit an interest in the foundations of mathematics were the ancient Greeks. Arithmetic or geometry Early Greek philosophy was dominated by a dispute as to which is more basic, arithmetic or geometry, and thus whether mathematics should be concerned primarily with the positive integers or the positive reals , the latter then being conceived as ratios of geometric quantities. The Greeks confined themselves to positive numbers , as negative numbers were introduced only much later in India by Brahmagupta. Underlying this dispute was a perceived basic dichotomy , not confined to mathematics but pervading all nature: The Pythagorean school of mathematics, founded on the doctrines of the Greek philosopher Pythagoras, originally insisted that only natural and rational numbers exist. The remarkable proof of this fact has been preserved by Aristotle. The contradiction between rationals and reals was finally resolved by Eudoxus of Cnidus , a disciple of Plato, who pointed out that two ratios of geometric quantities are equal if and only if they partition the set of positive rationals in the same way, thus anticipating the German mathematician Richard Dedekind â€" , who defined real numbers as such partitions. Being versus becoming Another dispute among pre-Socratic philosophers was more concerned with the physical world. Parmenides claimed that in the real world there is no such thing as change and that the flow of time is an illusion , a view with parallels in the Einstein-Minkowski four-dimensional space-time model of the universe. Heracleitus , on the other hand, asserted that change is all-pervasive and is reputed to have said that one cannot step into the same river twice. Zeno of Elea , a follower of Parmenides, claimed that change is actually impossible and produced four paradoxes to show this. The most famous of these describes a race between Achilles and a tortoise. Since Achilles can run much faster than the tortoise, let us say twice as fast, the latter is allowed a head start of one mile. When Achilles has run one mile, the tortoise will have run half as far againâ€"that is, half a mile. When Achilles has covered that additional half-mile, the tortoise will have run a further quarter-mile. So how can Achilles ever catch up with the tortoise see figure? Mathematically speaking, his argument involves the sum of the infinite geometric progression no finite partial sum of which adds up to 2. As Aristotle would later say, this progression is only potentially infinite. Universals The Athenian philosopher Plato believed that mathematical entities are not just human inventions but have a real existence. For instance, according to Plato, the number 2 is an ideal object. The number 2 is to be distinguished from a collection of two stones or two apples or, for that matter, two platinum balls in Paris. What, then, are these Platonic ideas? Already in ancient Alexandria some people speculated that they are words. Three dominant views prevailed: It would seem that Plato believed in a notion of truth independent of the human mind. The axiomatic method Perhaps the most important contribution to the foundations of mathematics made by the ancient Greeks was the axiomatic method and the notion of proof. This notion survives today, except for some cosmetic changes. The idea is this: It may take considerable ingenuity to discover a proof; but it is now held that it must be possible to check mechanically, step by step, whether a purported proof is indeed correct, and nowadays a computer should be able to do this. The mathematical statements that can be proved are called theorems , and it follows that, in principle, a mechanical device, such as a modern computer, can generate all theorems. Two questions about the axiomatic method were left unanswered by the ancients: Since the middle of the 20th century a gradually changing group of mostly French mathematicians under the pseudonym Nicolas Bourbaki has tried to emulate Euclid in writing a new Elements of Mathematics based on their theory of structures. Unfortunately, they just missed out on the new ideas from category theory. Number systems While the ancient Greeks were familiar with the positive integers, rationals, and reals, zero used as an actual number instead of denoting a missing number and the negative numbers were first used in India, as far as is known, by Brahmagupta in the 7th century ce. Much later, the German mathematician Carl Friedrich Gauss â€" proved the fundamental theorem of algebra , that all equations with complex coefficients have complex solutions, thus removing the principal motivation for introducing new numbers. Still, the Irish mathematician Sir William Rowan Hamilton â€"65 and the French mathematician Olinde Rodrigues â€" invented quaternions in the midth century, but these proved to be less popular in the scientific community until quite recently. Currently, a logical presentation of the number system, as taught at the university level, would be as follows: Here the letters, introduced by Nicolas Bourbaki, refer to the natural numbers, integers, rationals, reals, complex numbers, and quaternions, respectively, and the arrows indicate inclusion of each number system into the next. However, as has been shown, the historical development proceeds differently: This is the development, up to R, which is often adhered to at the high-school level. The reexamination of infinity Calculus reopens foundational questions Although mathematics flourished after the end of the Classical Greek period for years in Alexandria and, after an interlude in India and the Islamic world , again in Renaissance Europe, philosophical questions concerning the foundations of mathematics were not raised until the invention of calculus and then not by mathematicians but by the philosopher George Berkeley â€" Sir Isaac Newton in England and Gottfried Wilhelm Leibniz in Germany had independently developed the calculus on a basis of heuristic rules and methods markedly deficient in logical justification. Berkeley , concerned over the deterministic and atheistic implications of philosophical mechanism, set out to reveal contradictions in the calculus in his influential book The Analyst; or, A Discourse Addressed to an Infidel Mathematician. May we not call them the ghosts of departed quantities? Whether they do not submit to authority, take things upon trust, and believe points inconceivable? Moreover, the notion of limit was then explained quite rigorously, in answer to such thinkers as Zeno and Berkeley. It was not until the middle of the 20th century that the logician Abraham Robinson â€"74 showed that the notion of infinitesimal was in fact logically consistent and that, therefore, infinitesimals could be introduced as new kinds of numbers. This led to a novel way of presenting the calculus, called nonstandard analysis , which has, however, not become as widespread and influential as it might have. Non-Euclidean geometries When Euclid presented his axiomatic treatment of geometry, one of his assumptions, his fifth postulate , appeared to be less obvious or fundamental than the others. As it is now conventionally formulated, it asserts that there is exactly one parallel to a given line through a given point. It was then seen that Euclid had described not the one true geometry but only one of a number of possible geometries. In plane elliptic geometry there are no parallels to a given line through a given point; it may be viewed as the geometry of a spherical surface on which antipodal points have been identified and all lines are great circles. This was not viewed as revolutionary. This geometry is more difficult to visualize, but a helpful model presents the hyperbolic plane as the interior of a circle, in which straight lines take the form of arcs of circles perpendicular to the circumference. Another way to distinguish the three geometries is to look at the sum of the angles of a triangle. Contrasting triangles in Euclidean, elliptic, and hyperbolic spaces. Riemannian geometry The discovery that there is more than one geometry was of foundational significance and contradicted the German philosopher Immanuel Kant â€" Kant had argued that there is only one true geometry, Euclidean, which is known to be true a priori by an inner faculty or intuition of the mind. For Kant, and practically all other philosophers and mathematicians of his time, this belief in the unassailable truth of Euclidean geometry formed the foundation and justification for further explorations into the nature of reality. This divorce from geometric intuition added impetus to later efforts to rebuild assurance of truth on the basis of logic. See below The quest for rigour. What then is the correct geometry for describing the space actually space-time we live in? It turns out to be none of the above, but a more general kind of geometry, as was first discovered by the German mathematician Bernhard Riemann â€" In the early 20th century, Albert Einstein showed, in the context of his general theory of relativity , that the true geometry of space is only approximately Euclidean. Einstein spent the last part of his life trying to extend this idea to the electromagnetic force, hoping to reduce all physics to geometry, but a successful unified field theory eluded him. Cantor In the 19th century, the German mathematician Georg Cantor â€" returned once more to the notion of infinity and showed that, surprisingly, there is not just one kind of infinity but many kinds. In particular, while the set N of natural numbers and the set of all subsets of N are both infinite, the latter collection is more numerous, in a way that Cantor made precise, than the former. He proved that N, Z, and Q all have the same size, since it is possible to put them into one-to-one correspondence with one another, but that R is bigger, having the same size as the set of all subsets of N. However, Cantor was unable to prove the so-called continuum hypothesis , which asserts that there is no set that is larger than N yet smaller than the set of its subsets. Page 1 of 2.

## Chapter 5 : foundations of mathematics | History & Facts | www.nxgvision.com

*Foundations of Mathematical Logic by Haskell B. Curry This book is a thoroughly documented and comprehensive account of the constructive theory of the first-order predicate calculus. This is a calculus that is central to modern mathematical logic and important for mathematicians, philosophers, and scientists whose work impinges upon logic.*

Underlying all this were the basic logical concepts: The modern notation owes more to the influence of the English logician Bertrand Russell [â€”] and the Italian mathematician Giuseppe Peano [â€”] than to that of Frege. For an extensive discussion of logic symbols and operations, see formal logic. While this definition, even if unnecessarily cumbersome, is legitimate classically, it is not permitted in intuitionistic logic see below. Logic had been studied by the ancients, in particular by Aristotle and the Stoic philosophers. Philo of Megara flourished c. Yet the intimate connection between logic and mathematics had to await the insight of 19th-century thinkers, in particular Frege. Russell illustrated this by what has come to be known as the barber paradox: A barber states that he shaves all who do not shave themselves. Who shaves the barber? To avoid these contradictions Russell introduced the concept of types , a hierarchy not necessarily linear of elements and sets such that definitions always proceed from more basic elements sets to more inclusive sets, hoping that self-referencing and circular definitions would then be excluded. Mention should also be made of the system of the American philosopher Willard Van Orman Quine â€” , which admits a universal set. Although type theory was greatly simplified by Alonzo Church and the American mathematician Leon Henkin â€” , it came into its own only with the advent of category theory see below. Foundational logic The prominence of logic in foundations led some people, referred to as logicists , to suggest that mathematics is a branch of logic. The concepts of membership and equality could reasonably be incorporated into logic, but what about the natural numbers? Kronecker had suggested that, while everything else was made by man, the natural numbers were given by God. The logicists, however, believed that the natural numbers were also man-made, inasmuch as definitions may be said to be of human origin. Both definitions require an extralogical axiom to make them workâ€”the axiom of infinity , which postulates the existence of an infinite set. Since the simplest infinite set is the set of natural numbers, one cannot really say that arithmetic has been reduced to logic. Most mathematicians follow Peano , who preferred to introduce the natural numbers directly by postulating the crucial properties of 0 and the successor operation S, among which one finds the principle of mathematical induction. The logicist program might conceivably be saved by a 20th-century construction usually ascribed to Church , though he had been anticipated by the Austrian philosopher Ludwig Wittgenstein â€” There are some type-theoretical difficulties with this construction, but these can be overcome if quantification over types is allowed; this is finding favour in theoretical computer science. Impredicative constructions A number of 19th-century mathematicians found fault with the program of reducing mathematics to arithmetic and set theory as suggested by the work of Cantor and Frege. For example, when proving that every bounded nonempty set X of real numbers has a least upper bound a, one proceeds as follows. For this purpose, it will be convenient to think of a real number , following Dedekind , as a set of rationals that contains all the rationals less than any element of the set. It would seem that to do ordinary analysis one requires impredicative constructions. Russell and Whitehead tried unsuccessfully to base mathematics on a predicative type theory; but, though reluctant, they had to introduce an additional axiom, the axiom of reducibility, which rendered their enterprise impredicative after all. However, the German-American mathematician Hermann Weyl â€” and the American mathematician Solomon Feferman have shown that impredicative arguments such as the above can often be circumvented and are not needed for most, if not all, of analysis. On the other hand, as was pointed out by the Italian computer scientist Giuseppe Longo born , impredicative constructions are extremely useful in computer science â€”namely, for producing fixpoints entities that remain unchanged under a given process. Nonconstructive arguments Another criticism of the Cantor-Frege program was raised by Kronecker , who objected to nonconstructive arguments, such as the following proof that there exist irrational numbers a and b such that ab is rational. The argument is nonconstructive, because it does not tell us which alternative holds, even though more powerful mathematics will, as was shown by the Russian mathematician Aleksandr

Osipovich Gelfond â€" But there are other classical theorems for which no constructive proof exists. An ordered set is said to be well-ordered if every nonempty subset has a least element. It had been shown by the German mathematician Ernst Zermelo â€" that every set can be well-ordered, provided one adopts another axiom, the axiom of choice , which says that, for every nonempty family of nonempty sets, there is a set obtainable by picking out exactly one element from each of these sets. This axiom is a fertile source of nonconstructive arguments. Intuitionistic logic The Dutch mathematician L. This is the principle of the excluded third or excluded middle , which asserts that, for every proposition p, either p or not p; and equivalently that, for every p, not not p implies p. Brouwer did not claim that the principle of the excluded third always fails, only that it may fail in the presence of infinite sets. For an infinite set A, there is no way in which such an inspection can be carried out. Although Brouwer himself felt that mathematics was language-independent, his disciple Arend Heyting â€" set up a formal language for first-order intuitionistic arithmetic. While it cannot be said that many practicing mathematicians have followed Brouwer in rejecting this principle on philosophical grounds, it came as a great surprise to people working in category theory that certain important categories called topoi singular: In consequence of this fact, a theorem about sets proved constructively was immediately seen to be valid not only for sets but also for sheaves , which, however, lie beyond the scope of this article. According to this view, which goes back to Aristotle, infinite sets do not exist, except potentially. In fact, it is precisely in the presence of infinite sets that intuitionists drop the classical principle of the excluded third. An even more extreme position, called ultrafinitism, maintains that even very large numbers do not exist, say numbers greater than 10  Other logics While intuitionistic logic is obtained from classical logic by dropping the principle of the excluded third, other logics have also been proposed, though none has had a comparable impact on the foundations of mathematics. One may mention many-valued , or multivalued, logics, which admit a finite number of truth-values; fuzzy logic , with an imprecise membership relationship though, paradoxically, a precise equality relation ; and quantum logic, where conjunction may be only partially defined and implication may not be defined at all. Perhaps more important have been various so-called substructural logics in which the usual properties of the deduction symbol are weakened: This formalization project made sense only if the syntax of mathematics was consistent, for otherwise every syntactical statement would be provable, including that which asserts the consistency of mathematics. Yet, formalism is not deadâ€"in fact, most pure mathematicians are tacit formalistsâ€"but the naive attempt to prove the consistency of mathematics in a weaker system had to be abandoned. While no one, except an extremist intuitionist, will deny the importance of the language of mathematics, most mathematicians are also philosophical realists who believe that the words of this language denote entities in the real world. Following the Swiss mathematician Paul Bernays â€" , this position is also called Platonism, since Plato believed that mathematical entities really exist. The set of theorems provable statements is effectively enumerable, by virtue of the notion of proof being decidable. The language is consistent. Barkley Rosser later showed, could be replaced by assuming consistency. It was partly inspired by an argument that supposedly goes back to the ancient Greeks and which went something like this: Epimenides says that all Cretans are liars ; Epimenides is a Cretan; hence Epimenides is a liar. No mathematician doubts assumption 1; by looking at a purported proof of a theorem, suitably formalized, it is possible for a mathematician, or even a computer, to tell whether it is a proof. By listing all proofs in, say, alphabetic order, an effective enumeration of all theorems is obtained. However, moderate intuitionists could draw a different conclusion, because they are not committed to assumption 2. Intuitionists have always believed that, for a statement to be true, its truth must be knowable. Moreover, moderate intuitionists might concede to formalists that to say that a statement is known to be true is to say that it has been proved. Still, some intuitionists do not accept the above argument. Other logicians are more skeptical and want to replace the notion of truth by that of truth in a model. Recursive definitions Peano had observed that addition of natural numbers can be defined recursively thus: This notion can easily be extended to subsets of Nk and, by a simple trick called arithmetization, to sets of strings of words in a language. It is not difficult to show that all primitive recursive functions can be calculated. But primitive recursive functions are not the only numerical functions that can be calculated. All recursive functions can be calculated with pencil and paper or, even more primitively, by moving pebbles calculi in Latin from one location to another, using some finite set of instructions, nowadays called a program. Conversely, only recursive functions can be so calculated, or computed by a theoretical machine introduced by the English mathematician Alan Turing â€”54 , or by a modern computer, for that matter. The Church-Turing thesis asserts that the informal notion of calculability is completely captured by the formal notion of recursive functions and hence, in theory, replicable by a machine. Church and Turing, while seeking an algorithmic mechanical test for deciding theoremhood and thus potentially deleting nontheorems, proved independently, in , that such an algorithmic method was impossible for the first-order predicate logic see logic, history of: The Church-Turing theorem of undecidability, combined with the related result of the Polish-born American mathematician Alfred Tarski â€”83 on undecidability of truth, eliminated the possibility of a purely mechanical device replacing mathematicians. Computers and proof While many mathematicians use computers only as word processors and for the purpose of communication, computer-assisted computations can be useful for discovering potential theorems. For example, the prime number theorem was first suggested as the result of extensive hand calculations on the prime numbers up to 3,, by the Swiss mathematician Leonhard Euler â€”83 , a process that would have been greatly facilitated by the availability of a modern computer. Computers may also be helpful in completing proofs when there are a large number of cases to be considered. The renowned computer-aided proof of the four-colour mapping theorem by the American mathematicians Kenneth Appel born and Wolfgang Haken born even goes beyond this, as the computer helped to determine which cases were to be considered in the next step of the proof. Yet, in principle, computers cannot be asked to discover proofs, except in very restricted areas of mathematicsâ€”such as elementary Euclidean geometryâ€”where the set of theorems happens to be recursive, as was proved by Tarski. As the result of earlier investigations by Turing, Church, the American mathematician Haskell Brooks Curry â€”82 , and others, computer science has itself become a branch of mathematics. Thus, in theoretical computer science, the objects of study are not just theorems but also their proofs, as well as calculations, programs, and algorithms. Theoretical computer science turns out to have a close connection to category theory. Although this lies beyond the scope of this article, an indication will be given below. Abstraction in mathematics One recent tendency in the development of mathematics has been the gradual process of abstraction. The Norwegian mathematician Niels Henrik Abel â€”29 proved that equations of the fifth degree cannot, in general, be solved by radicals. These concrete groups soon gave rise to abstract groups, which were described axiomatically. Then it was realized that to study groups it was necessary to look at the relation between different groupsâ€”in particular, at the homomorphisms which map one group into another while preserving the group operations. Thus people began to study what is now called the concrete category of groups, whose objects are groups and whose arrows are homomorphisms. It did not take long for concrete categories to be replaced by abstract categories, again described axiomatically. A category has not only objects but also arrows referred to also as morphisms, transformations, or mappings between them. Many categories have as objects sets endowed with some structure and arrows, which preserve this structure. Thus, there exist the categories of sets with empty structure and mappings, of groups and group-homomorphisms, of rings and ring-homomorphisms, of vector spaces and linear transformations, of topological spaces and continuous mappings, and so on. There even exists, at a still more abstract level, the category of small categories and functors , as the morphisms between categories are called, which preserve relationships among the objects and arrows. Not all categories can be viewed in this concrete way. For example, the formulas of a deductive system may be seen as objects of a category whose arrows f: In fact, this point of view is important in theoretical computer science, where formulas are thought of as types and deductions as operations. More formally, a category consists of 1 a collection of objects A, B, C,. Additionally, the associative law and the identities are required to hold where the compositions are defined â€”i. In a sense, the objects of an abstract category have no windows, like the monads of Leibniz. To infer the interior of an object A one need only look at all the arrows from other objects to A.

## Chapter 6 : Foundations of mathematics - The quest for rigour | www.nxgvision.com

*Mathematical logic is a subfield of mathematics exploring the applications of formal logic to mathematics. It bears close connections to metamathematics, the foundations of mathematics, and theoretical computer science. [1].*

The challenge takes the following form. There exist infinitely many ways of identifying the natural numbers with pure sets. Let us restrict, without essential loss of generality, our discussion to two such ways: Which of these consists solely of true identity statements: It seems very difficult to answer this question. It is not hard to see how a successor function and addition and multiplication operations can be defined on the number-candidates of I and on the number-candidates of II so that all the arithmetical statements that we take to be true come out true. Indeed, if this is done in the natural way, then we arrive at isomorphic structures in the set-theoretic sense of the word , and isomorphic structures make the same sentences true they are elementarily equivalent. So it is impossible that both accounts are correct. If both accounts were correct, then the transitivity of identity would yield a purely set theoretic falsehood. Summing up, we arrive at the following situation. On the one hand, there appear to be no reasons why one account is superior to the other. On the other hand, the accounts cannot both be correct. The proper conclusion to draw from this conundrum appears to be that neither account I nor account II is correct. Since similar considerations would emerge from comparing other reasonable-looking attempts to reduce natural numbers to sets, it appears that natural numbers are not sets after all. It is clear, moreover, that a similar argument can be formulated for the rational numbers, the real numbersâ€¦ Benacerraf concludes that they, too, are not sets at all. A platonist can uphold the claim that the natural numbers can be embedded into the set-theoretic universe while maintaining that the embedding should not be seen as an ontological reduction. But then it seems that platonists would have to take a similar line with respect to the rational numbers, the complex numbers, â€¦. Whereas maintaining that the natural numbers are sui generis admittedly has some appeal, it is perhaps less natural to maintain that the complex numbers, for instance, are also sui generis. And, anyway, even if the natural numbers, the complex numbers, â€¦ are in some sense not reducible to anything else, one may wonder if there may not be another way to elucidate their nature. Roughly, non-algebraic theories are theories which appear at first sight to be about a unique model: We have seen examples of such theories: But his challenge does not apply to algebraic theories. Algebraic theories are not interested in mathematical objects per se; they are interested in structural aspects of mathematical objects. This led Benacerraf to speculate whether the same could not be true also of non-algebraic theories. Shapiro and Resnik hold that all mathematical theories, even non-algebraic ones, describe structures. This position is known as structuralism Shapiro ; Resnik  Structures consists of places that stand in structural relations to each other. Thus, derivatively, mathematical theories describe places or positions in structures. But they do not describe objects. The number three, for instance, will on this view not be an object but a place in the structure of the natural numbers. Systems are instantiations of structures. The systems that instantiate the structure that is described by a non-algebraic theory are isomorphic with each other, and thus, for the purposes of the theory, equally good. The systems I and II that were described in section 4. But neither are the number three. For the number three is an open place in the natural number structure, and this open place does not have any internal structure. Systems typically contain structural properties over and above those that are relevant for the structures that they are taken to instantiate. Sensible identity questions are those that can be asked from within a structure. They are those questions that can be answered on the basis of structural aspects of the structure. Identity questions that go beyond a structure do not make sense. The question mixes two different structures: Even if there were no infinite systems to be found in Nature, the structure of the natural numbers would exist. Thus structures as Shapiro understands them are abstract, platonic entities. In textbooks on set theory we also find a notion of structure. But this cannot be the notion of structure that structuralism in the philosophy of mathematics has in mind. For the set theoretic notion of structure presupposes the concept of set, which, according to structuralism, should itself be explained in structural terms. Or, to put the point differently, a set-theoretical structure is merely a system that instantiates a structure that is ontologically prior to it. Nonetheless, the motivation for extending ante rem structuralism even to the most encompassing mathematical discipline set theory is not entirely evident Burgess  For set theory, it seems hard to mount an identification challenge: It appears that ante rem structuralism describes the notion of a structure in a somewhat circular manner. A structure is described as places that stand in relation to each other, but a place cannot be described independently of the structure to which it belongs. Yet this is not necessarily a problem. For the ante rem structuralist, the notion of structure is a primitive concept, which cannot be defined in other more basic terms. At best, we can construct an axiomatic theory of mathematical structures. Structures and places in structures may not be objects, but they are abstract. So it is natural to wonder how we succeed in obtaining knowledge of them. This problem has been taken by certain philosophers as a reason for developing a nominalist theory of mathematics and then to reconcile this theory with basic tenets of structuralism. The nominalistic reconstruction of scientific theories proved to be a difficult task. Quine, for one, abandoned it after this initial attempt. In the past decades many theories have been proposed that purport to give a nominalistic reconstruction of mathematics. In a nominalist reconstruction of mathematics, concrete entities will have to play the role that abstract entities play in platonistic accounts of mathematics, and concrete relations such as the part-whole relation have to be used to simulate mathematical relations between mathematical objects. But here problems arise. First, already Hilbert observed that, given the discretization of nature in quantum mechanics, the natural sciences may in the end claim that there are only finitely many concrete entities Hilbert  Yet it seems that we would need infinitely many of them to play the role of the natural numbers â€” never mind the real numbers. Where does the nominalist find the required collection of concrete entities? Field made an earnest attempt to carry out a nominalistic reconstruction of Newtonian mechanics Field  The basic idea is this. Field wanted to use concrete surrogates of the real numbers and functions on them. He adopted a realist stance toward the spatial continuum, and took regions of space to be as physically real as chairs and tables. And he took regions of space to be concrete after all, they are spatially located. If we also count the very disconnected ones, then there are as many regions of Newtonian space as there are subsets of the real numbers. And then there are enough concrete entities to play the role of the natural numbers, the real numbers, and functions on the real numbers. And the theory of the real numbers and functions on them is all that is needed to formulate Newtonian mechanics. Of course it would be even more interesting to have a nominalistic reconstruction of a truly contemporary scientific theory such as Quantum Mechanics. But given that the project can be carried out for Newtonian mechanics, some degree of initial optimism seems justified. This project clearly has its limitations. It may be possible nominalistically to interpret theories of function spaces on the real numbers, say. But it seems far-fetched to think that along Fieldian lines a nominalistic interpretation of set theory can be found. For it would mean that, to some extent at least, mathematical entities appear to be dispensable after all. He would thereby have taken an important step towards undermining the indispensability argument for Quinean modest platonism in mathematics, for, to some extent, mathematical entities appear to be dispensable after all. This leads to a position that has been called ultra-finitism Essenin-Volpin  On most accounts, ultra-finitism leads, like intuitionism, to revisionism in mathematics. For it would seem that one would then have to say that there is a largest natural number, for instance. From the outside, a theory postulating only a finite mathematical universe appears proof-theoretically weak, and therefore very likely to be consistent. But Woodin has developed an argument that purports to show that from the ultra-finitist perspective, there are no grounds for asserting that the ultra-finitist theory is likely to be consistent Woodin  Regardless of this argument the details of which are not discussed here , many already find the assertion that there is a largest number hard to swallow. But Lavine has articulated a sophisticated form of set-theoretical ultra-finitism which is mathematically non-revisionist Lavine  He has developed a detailed account of how the principles of ZFC can be taken to be principles that describe determinately finite sets, if these are taken to include indefinitely large ones. Admittedly it is not a simple task to give an account of how humans obtain knowledge of spacetime regions. But at least according to many but not all philosophers spacetime regions are physically real. So we are no longer required to explicate how flesh and blood mathematicians stand in contact with non-physical entities. This leads to versions of nominalist structuralism, which can be outlined as follows. Let us focus on mathematical analysis. The nominalist structuralist denies that any concrete physical system is the unique intended interpretation of analysis. All concrete physical systems that satisfy the basic principles of Real Analysis RA would do equally well. This entails that, as with ante rem structuralism, only structural aspects are relevant to the truth or falsehood of mathematical statements. But unlike ante rem structuralism, no abstract structure is postulated above and beyond concrete systems. According to in rebus structuralism, no abstract structures exist over and above the systems that instantiate them; structures exist only in the systems that instantiate them. Nominalist structuralism is a form of in rebus structuralism. But in rebus structuralism is not exhausted by nominalist structuralism.

## Chapter 7 : Search results for ` Mathematical Logic and Foundations` - PhilPapers

*Carnap's work in the foundations of logic and mathematics from a contem- porary perspective, in light of what we now know about the foundations of mathematics and (to a lesser degree) the foundations of physics.*

## Chapter 8 : Foundations of Mathematics

*the Foundations of Mathematics should give a precise deï¬•nition of what a mathematical statement is and what a mathematical proof is, as we do in Chapter II, which covers model theory and proof theory.*

## Chapter 9 : Foundations of Mathematical Logic

*Second-Order Logic and Foundations of Mathematics- by Jouko Vaananen more Informal reasoning, formal languages of the first-order and second-order logics, semantics Second and Higher-Order Logic - by Robert Harper intro.*